The total bondage number of grid graphs [*]

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Abstract: The total domination number of a graph G without isolated vertices is the minimum number of vertices that dominate all vertices in G. The total bondage number $b_t(G)$ of G is the minimum number of edges whose removal enlarges the total domination number. This paper considers grid graphs. An (n, m)-grid graph $G_{n,m}$ is defined as the cartesian product of two paths P_n and P_m. This paper determines the exact values of $b_t(G_{n,2})$ and $b_t(G_{n,3})$, and establishes some upper bounds of $b_t(G_{n,4})$.

Keywords: total dominating set, total domination number, total bondage number, grid graphs.
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1   Introduction

For notation and graph-theoretical terminology not defined here we follow [27]. Specifically, let $G = (V, E)$ be an undirected graph without loops and multi-edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set, which is a subset of $\{xy|\ xy$ is an unordered pair of $V\}$. A graph G is *nonempty* if $E(G) \neq \emptyset$. Two vertices x and y are *adjacent* if $xy \in E(G)$. For a vertex x, we call the vertices adjacent to it the *neighbors* of x. We use P_n and C_n to denote a path and a cycle of order n throughout this paper.

A subset $D \subseteq V(G)$ is called a *dominating set* of G if every vertex not in D has at least one neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets.

The domination is so an important and classic conception that it has become one of the most widely studied topics in graph theory, and also is frequently used to study properties of interconnection networks. The early results on this subject have been surveyed and detailed in the two excellent domination books by Haynes, Hedetniemi, and Slater [11, 12]. In the recent decade, a large number of research papers on domination as well as related topics appear in many scientific journals because of their applications in many fields such as networks and so on.

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A dominating set D of a graph G without isolated vertices is called to be *total* if every vertex in G has at least one neighbor in D. The minimum cardinality among all total dominating sets is called the *total domination number* of G, denoted by $\gamma_t(G)$. It is clear that $\gamma(G) \leqslant \gamma_t(G) \leqslant 2\gamma(G)$ for any graph G without isolated vertices.

The total domination in graphs was introduced by Cockayne *et al.* [1] in 1980. The total domination in graphs has been extensively studied in the literature. In 2009, Henning [13] gave a survey of selected recent results on this topic.

In 1990, Fink *et al.* [4] introduced the bondage number as a parameter for measuring the vulnerability of the interconnection network under link failure. The minimum dominating set of sites plays an important role in the network for it dominates the whole network with the minimum cost. So we must consider whether its function remains good under attack. Suppose that someone such as a saboteur does not know which sites in the network take part in the dominating role, but does know that the set of these special sites corresponds to a minimum dominating set in the related graph. Then how many links does he have to attack so that the cost can not remains the same in order to dominate the whole network? That minimum number of links is just the bondage number.

The *bondage number* $b(G)$ of a nonempty graph G is the minimum number of edges whose removal from G results in a graph with larger domination number than $\gamma(G)$. Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$, the bondage number of a nonempty graph is well defined. Many results on this topic are obtained in the literature. The exact values of the bondage numbers for some graphs are determined, for example, a complete graph, a path, a cycle, a complete t-partite graph [4], a tree [7,8,25,26], for the cartesian product of two cycles $C_4 \times C_n$ [19] and $C_3 \times C_n$ [23], and for other graphs [15, 17, 18]. Some upper bounds of the bondage numbers for graphs are established, see, for example, [3, 4, 9, 10, 16, 18, 22, 25] for general graphs, [2, 5, 20] for planar graphs. In particular, very recently, Hu and Xu [14] have showed that the problem determining bondage number for general graphs is NP-hard.

Following Fink et al., Kulli and Patwari [21] proposed the concept of the total bondage number for a graph. The total bondage number $b_t(G)$ of a graph G is the minimum number of edges whose removal results in a graph with total domination number larger than $\gamma_t(G)$. If $b_t(G)$ does not exist, for example a star graph $K_{1,n}$, we define $b_t(G) = \infty$. Kulli and Patwari [21] calculated the exact values of $b_t(G)$ for some standard graphs such as a cycle C_n and a path P_n for $n \geqslant 4$, a complete bipartite graph $K_{m,n}$ and a complete K_n. Sridharan *et al.* [24] showed that for any positive integer k there exists a tree T with $b_t(T) = k$. These authors also established the upper bounds of $b_t(G)$ for a graph G in terms of its order. As far as we know, no much research work on the total bondage number was reported in the literature except for the above-mentioned. However, Hu and Xu [14] also showed that the problem determining total bondage number for general graphs is NP-hard.

An (n, m)-grid graph $G_{n,m}$ is the cartesian product $P_n \times P_m$ of two paths P_n and P_m. In this paper, we consider $b_t(G_{n,m})$. Since $G_{1,m} \cong P_m$, we assume $n \geq 2$ under our discussion. In 2002, Gravier [6] determined $\gamma_t(G_{n,m})$ for any $m \in \{1, 2, 3, 4\}$, based on

which we obtain the following results.

$$b_t(G_{n,2}) = \begin{cases} 1 \text{ if } n \equiv 0 \,(\text{mod}\,3), \\ 2 \text{ if } n \equiv 2 \,(\text{mod}\,3), \\ 3 \text{ if } n \equiv 1 \,(\text{mod}\,3); \end{cases}$$
$$b_t(G_{n,3}) = 1; \; b_t(G_{6,4}) = 2, \text{and}$$
$$b_t(G_{n,4}) \begin{cases} = 1 & \text{if } n \equiv 1 \,(\text{mod}\,5) \text{ and } n \neq 6; \\ = 2 & \text{if } n \equiv 4 \,(\text{mod}\,5); \\ \leq 3 & \text{if } n \equiv 2 \,(\text{mod}\,5); \\ \leq 4 & \text{if } n \equiv 0, 3 \,(\text{mod}\,5). \end{cases}$$

The proofs of these results are in Section 3, Section 4 and Section 5, respectively. In Section 2, we give two preliminary results to be used in our proofs.

2 Preliminary results

Throughout this paper, we assume that a path P_n has the vertex-set $V(P_n) = \{1, \cdots, n\}$. An (n, m)-grid graph $G_{n,m}$ is defined as the Cartesian product $G_{n,m} = P_n \times P_m$ with vertex-set $V(G_{n,m}) = \{x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and two vertices x_{ij} and $x_{i'j'}$ being linked by an edge if and only if either $i = i' \in V(P_n)$ and $jj' \in E(P_m)$, such an edge is called a *vertical edge*, or $j = j' \in V(P_m)$ and $ii' \in E(P_n)$, such an edge is called a *horizontal edge*. The graph shown in Figure 1 is a $(4, 3)$-grid graph $G_{4,3}$. It is clear, as a graphic operation, that the cartesian product satisfies commutative associative law if identify isomorphic graphs, that is, $G_{n,m} \cong G_{m,n}$.
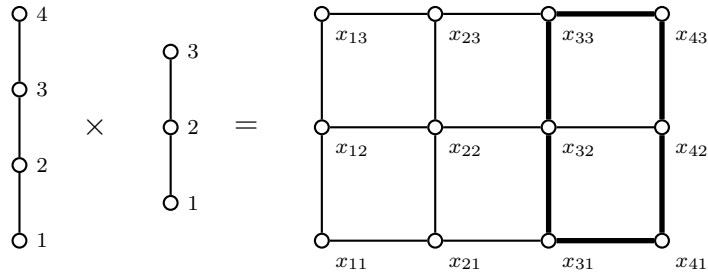

Figure 1: A $(4, 3)$-grid graph $G_{4,3} = P_4 \times P_3$

The following notations continually appear in our proofs. For a given integer t with $t < n$, $G_{t,m}$ is a subgraph of $G_{n,m}$. We use the notation $H_{n-t,m}$ to denote $G_{n,m} - G_{t,m}$, that is, $H_{n-t,m}$ is a subgraph of $G_{n,m}$ induced by the set of vertices $\{x_{ij} \mid t + 1 \leq i \leq n, 1 \leq j \leq m\}$. Clearly, $H_{n-t,m} \cong G_{n-t,m}$. For example, the graph shown in Figure 1 by heavy lines is a subgraph $H_{2,3}$ of $G_{4,3}$, where $n = 4, t = 2$ and $m = 3$.

Note that both $G_{0,m}$ and $H_{n-n,m}$ are nominal graphs. For convenience of statements, we allow $G_{0,m}$ and $H_{n-n,m}$ to appear in our proofs. If so, we specify their total dominating sets are empty.

In Addition, let $Y_i = \{x_{ij} \mid 1 \leq j \leq m\}$ for $1 \leq i \leq n$, called a set of *vertical vertices* in $G_{n,m}$.

We state some useful results on $\gamma_t(G_{n,m})$ to be used in our proofs.

Lemma 2.1 (Gravier [6]) *Let n be a positive integer. Then*
$\gamma_t(G_{n,2}) = 2\lfloor \frac{n+2}{3} \rfloor$ *for $n \geq 1$;*
$\gamma_t(G_{1,3}) = 2$ *and* $\gamma_t(G_{n,3}) = n$ *for $n \geq 2$;*
$$\gamma_t(G_{n,4}) = \begin{cases} \lfloor \frac{6n+8}{5} \rfloor & \text{if } n \equiv 1, 2, 4 \,(\text{mod } 5), \\ \lfloor \frac{6n+8}{5} \rfloor + 1 & \text{otherwise} \end{cases} \quad \textit{for } n \geq 4.$$

Lemma 2.2 *Let D be a total dominating set of $G_{n,m}$. Then $\gamma_t(G_{i,m}) \leq |D \cap V(G_{i+1,m})|$ for $1 \leq i \leq n-1$ and $m \geq 2$.*

Proof. Let $D' = D \cap V(G_{i+1,m})$. If $i = 1$, then the lemma holds clearly. Assume $i \geq 2$ below.

If $D' \cap Y_{i+1} = \emptyset$, then D' is a total dominating set of $G_{i,m}$, and hence $\gamma_t(G_{i,m}) \leq |D'|$.

Assume $D' \cap Y_{i+1} \neq \emptyset$ below. By the definition of a total dominating set, it is possible that a vertex in $D' \cap Y_i$ is dominated only by vertices $D' \cap Y_{i+1}$. Let A_i be the set of vertices in $D' \cap Y_i$ dominated only by vertices in $D' \cap Y_{i+1}$, and let $B_i = \{j|\ x_{ij} \in A_i\}$. Then $D'' = (D' \backslash Y_{i+1}) \cup \{x_{(i-1)j}|\ j \in B_i\}$ is a total dominating set of $G_{i,m}$ and $|D''| \leq |D'|$. Thus, we have $\gamma_t(G_{i,m}) \leq |D''| \leq |D'|$. The lemma follows. ∎

Lemma 2.3 (Kulli and Patwari [21]) *For a path P_n with $n \geqslant 4$,*

$$b_t(P_n) = \begin{cases} 2 & \text{if } n \equiv 2 \,(\text{mod } 4); \\ 1 & \text{otherwise.} \end{cases}$$

Since $G_{1,m} \cong P_m$ and $G_{n,1} \cong P_n$, by Lemma 2.3, we assume that if one of n and m is 1, then the other is at least 4 when we consider the existence of $b_t(G_{n,m})$.

3 The total bondage number of $G_{n,2}$

In this section, we determine the exact value of $b_t(G_{n,2})$ for $n \geq 2$. Since the computation of $b_t(G_{n,2})$ strongly depends on the value of $\gamma_t(G_{n,2})$ in Lemma 2.1, the process of our proofs consists of several lemmas according to the value of n modulo 3.

Lemma 3.1 $b_t(G_{n,2}) \leq 2$ *for $n \not\equiv 1 \,(\text{mod } 3)$.*

Proof. By Lemma 2.1, we have

$$\gamma_t(G_{n-1,2}) = \gamma_t(G_{n,2}) \text{ if } n \not\equiv 1 \,(\text{mod } 3). \tag{3.1}$$

Let $B = \{x_{(n-1)1}x_{n1}, x_{(n-1)2}x_{n2}\} \subset E(G_{n,2})$, and let $H = G_{n,2} - B$. By (3.1), we have

$$\gamma_t(H) = 2 + \gamma_t(G_{n-1,2}) \geq 1 + \gamma_t(G_{n,2}),$$

which implies that $b_t(G_{n,2}) \leq |B| = 2$. ∎

Lemma 3.2 *If $n \equiv 1 \,(\text{mod } 3)$, then $\gamma_t(G_{n,2} - x_{nj}) = \gamma_t(G_{n,2}) - 1$ for each $j = 1, 2$.*

Proof. Without loss of generality, we only consider the case $j = 1$. By the hypothesis, $n \geq 4$. It can be direct check that the lemma holds for $n = 4$. Assume $n \geq 7$ below. Let D be a minimum total dominating set of $G_{n,2} - x_{n1}$. We need to show $|D| = \gamma_t(G_{n,2}) - 1$.

We consider a subgraph $G_{n-4,2}$ and let D' be a minimum total dominating set of $G_{n-4,2}$. By Lemma 2.1, $|D'| = 2\lfloor \frac{n-4+2}{3} \rfloor = 2\lfloor \frac{n-2}{3} \rfloor$. Clearly, $D' \cup \{x_{(n-1)2}, x_{(n-2)2}, x_{(n-3)2}\}$ is a total dominating set of $G_{n,2} - x_{n1}$. Since $n \equiv 1 \,(\mathrm{mod}\, 3)$, we have $2\lfloor \frac{n+4}{3} \rfloor = 2\lfloor \frac{n+2}{3} \rfloor = \gamma_t(G_{n,2})$ by Lemma 2.1. It follows that

$$
\begin{aligned}
|D| &\le |D'| + 3 = 2\lfloor \tfrac{n-2}{3} \rfloor + 3 \\
&= 2\lfloor \tfrac{n+4}{3} \rfloor - 1 = 2\lfloor \tfrac{n+2}{3} \rfloor - 1 \\
&= \gamma_t(G_{n,2}) - 1,
\end{aligned}
$$

that is,

$$
|D| \le \gamma_t(G_{n,2}) - 1. \tag{3.2}
$$

We now prove that $|D| \ge \gamma_t(G_{n,2}) - 1$. If one of $x_{(n-1)1}$ and x_{n2} belongs to D, then D is a total dominating set of $G_{n,2}$. By (3.2), we can deduce a contradiction as follows. $\gamma_t(G_{n,2}) \le |D| \le \gamma_t(G_{n,2}) - 1$. It follows that neither of $x_{(n-1)1}$ and x_{n2} belongs to D. Since D is a total dominating set of $G_{n,2} - x_{n1}$, the vertex $x_{(n-1)2}$ must be in D to dominate x_{n2}. Thus $D \cup \{x_{(n-1)1}\}$ is a total dominating set of $G_{n,2}$, and so

$$
|D| = |D \cup \{x_{(n-1)1}\}| - 1 \ge \gamma_t(G_{n,2}) - 1.
$$

The lemma follows. ∎

Lemma 3.3 $b_t(G_{n,2}) = 1$ for $n \equiv 0 \,(\mathrm{mod}\, 3)$.

Proof. We only need to show

$$
\gamma_t(G_{n,2} - x_{(n-1)1}x_{n1}) \ge \gamma_t(G_{n,2}) + 1. \tag{3.3}
$$

Let $H = G_{n,2} - x_{(n-1)1}x_{n1}$ and D be a minimum total dominating set of H. Then the vertex x_{n2} must be in D otherwise D can not dominate the vertex x_{n1} in H. Moreover, D is either a total dominating set of $G_{n+1,2}$ if x_{n1} is in D or a total dominating set of $G_{n+1,2} - x_{(n+1)1}$ if x_{n1} is not in D. Since $n \equiv 0 \,(\mathrm{mod}\, 3)$, we have $n + 1 \equiv 1 \,(\mathrm{mod}\, 3)$ and $\lfloor \frac{n}{3} \rfloor = \lfloor \frac{n+2}{3} \rfloor$. By Lemma 2.1 and Lemma 3.2, we have

$$
\begin{aligned}
\gamma_t(H) = |D| &\ge \gamma_t(G_{n+1,2}) - 1 = 2\lfloor \tfrac{n+1+2}{3} \rfloor - 1 = 2\lfloor \tfrac{n}{3} \rfloor + 1 \\
&= 2\lfloor \tfrac{n+2}{3} \rfloor + 1 = \gamma_t(G_{n,2}) + 1.
\end{aligned}
$$

The lemma follows. ∎

Lemma 3.4 $b_t(G_{n,2}) = 2$ for $n \equiv 2 \,(\mathrm{mod}\, 3)$.

Proof. To prove the lemma, we only need to show $b_t(G_{n,2}) \ge 2$ by Lemma 3.1. To this end, we only need to show $\gamma_t(G_{n,2} - e) = \gamma_t(G_{n,2})$ for any edge e in $G_{n,2}$. Let e be any edge in $G_{n,2}$. We only need to prove that $\gamma_t(G_{n,2} - e) \le \gamma_t(G_{n,2})$ since $\gamma_t(G_{n,2}) \le \gamma_t(G_{n,2} - e)$ clearly. We attain this aim by constructing a total dominating set D of $G_{n,2} - e$ such that $|D| = 2\lfloor \frac{n+2}{3} \rfloor$, which means $|D| = \gamma_t(G_{n,2})$ by Lemma 2.1.

We consider two cases according as that e is vertical or horizontal.

Suppose that e is a vertical edge $e = x_{i1}x_{i2}$, where $1 \le i \le n$. Let

$$
D = \begin{cases} \{x_{kj} : k \equiv 1 \,(\mathrm{mod}\, 3), j = 1, 2\} & \text{if } i \equiv 2 \,(\mathrm{mod}\, 3); \\ \{x_{kj} : k \equiv 2 \,(\mathrm{mod}\, 3), j = 1, 2\} & \text{otherwise.} \end{cases}
$$

Then D is a total dominating set of $G_{n,2} - e$ and $|D| = 2\left\lfloor\frac{n+2}{3}\right\rfloor$.

Suppose now that e is a horizontal edge, maybe $e = x_{i1}x_{(i+1)1}$ or $e = x_{i2}x_{(i+1)2}$, where $1 \leq i \leq n-1$. Without loss of generality, set $e = x_{i1}x_{(i+1)1}$. We consider two subcases to construct D, respectively.

Assume $i \not\equiv 1 \pmod 3$. Let

$$D = \begin{cases} \{x_{kj} : k \equiv 2 \pmod 3, j = 1, 2\} & \text{if } i \equiv 0 \pmod 3; \\ \{x_{kj} : k \equiv 1 \pmod 3, j = 1, 2\} & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

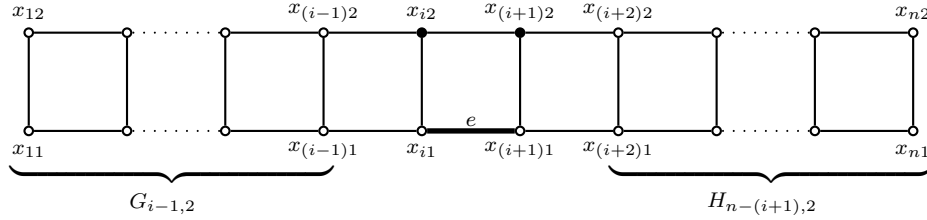Then D is a total dominating set of $G_{n,2} - e$ and $|D| = 2\lfloor\frac{n+2}{3}\rfloor$.

Figure 2: Two subgraphs $G_{i-1,2}$ and $H_{n-(i+1),2}$ of $G_{n,2}$

Assume now $i \equiv 1 \pmod 3$. We consider $G_{i-1,2}$ and $H_{n-(i+1),2}$ (see Figure 2). Let D' and D'' be minimum total dominating sets of $G_{i-1,2}$ and $H_{n-(i+1),2}$, respectively. Then $D = D' \cup D'' \cup \{x_{i2}, x_{(i+1)2}\}$ is a total dominating set of $G_{n,2} - e$. Note $D' = \emptyset$ if $i = 1$ and $D'' = \emptyset$ if $i = n-1$. Thus, by Lemma 2.1,

$$\begin{aligned} |D| &= |D'| + |D''| + 2 \\ &= 2\left\lfloor\frac{i-1+2}{3}\right\rfloor + 2\left\lfloor\frac{n-i-1+2}{3}\right\rfloor + 2 \\ &= 2\left\lceil\frac{n+2}{3}\right\rceil. \end{aligned}$$

The lemma follows. ∎

Lemma 3.5 $b_t(G_{n,2}) = 3$ for $n \equiv 1 \pmod 3$.

Proof. Since $n \equiv 1 \pmod 3$, $n-1 \equiv 0 \pmod 3$. By (3.3), for the edge $e_0 = x_{(n-2)1}x_{(n-1)1}$, we have

$$\gamma_t(G_{n-1,2} - e_0) \geq \gamma_t(G_{n-1,2}) + 1. \tag{3.4}$$

Choose other two edges e_1, e_2 in $G_{n,2}$, where $e_1 = x_{(n-1)1}x_{n1}$ and $e_2 = x_{(n-1)2}x_{n2}$. Let $H = G_{n,2} - \{e_0, e_1, e_2\}$. Then $H = (G_{n-1,2} - e_0) + H_{n-(n-1),2}$ and any total dominating set of H must contain vertices x_{n1} and x_{n2}. By (3.4) and Lemma 2.1, we have

$$\begin{aligned} \gamma_t(H) &= \gamma_t(G_{n-1,2} - e_0) + 2 \\ &\geq \gamma_t(G_{n-1,2}) + 1 + 2 \\ &= 2\left\lfloor\frac{n-1+2}{3}\right\rfloor + 3 \\ &= 2\left\lfloor\frac{n+4}{3}\right\rfloor + 1 = 2\left\lfloor\frac{n+2}{3}\right\rfloor + 1 \\ &= \gamma_t(G_{n,2}) + 1, \end{aligned}$$

which implies $b_t(G_{n,2}) \leq 3$.

Now we prove $b_t(G_{n,2}) \geq 3$. To the end, let e_1 and e_2 be any two edges in $G_{n,2}$, and $H = G_{n,2} - \{e_1, e_2\}$. We only need to prove $\gamma_t(H) \leq \gamma_t(G_{n,2})$. We consider three cases.

Case 1 Both e_1 and e_2 are vertical edges.

Let $e_1 = x_{i1}x_{i2}$, $e_2 = x_{j1}x_{j2}$, $i < j$, and let

$$D = \begin{cases} \{x_{kl} : k \equiv 1 \,(\mathrm{mod}\,3), l = 1, 2\} & \text{if } i, j \not\equiv 1 \,(\mathrm{mod}\,3); \\ \{x_{kl} : k \equiv 2 \,(\mathrm{mod}\,3), l = 1, 2\} \cup \{x_{(n-1)1}, x_{(n-1)2}\} & \text{if } i, j \not\equiv 2 \,(\mathrm{mod}\,3); \\ \{x_{kl} : k \equiv 0 \,(\mathrm{mod}\,3), l = 1, 2\} \cup \{x_{21}, x_{22}\} & \text{otherwise.} \end{cases}$$

Then D is a total dominating set of H and $\gamma_t(H) \leq |D| = 2\left\lfloor \frac{n+2}{3} \right\rfloor$. By Lemma 2.1, $|D| = \gamma_t(G_{n,2})$. Thus, for two vertical edges e_1 and e_2, we have

$$\gamma_t(H) = \gamma_t(G_{n,2} - \{e_1, e_2\}) \leq \gamma_t(G_{n,2}). \tag{3.5}$$

Case 2 One of e_1 and e_2 is horizontal and the other is vertical.

Without loss of generality, suppose that e_1 is horizontal and e_2 is vertical, and let $e_1 = x_{i1}x_{(i+1)1}$ and $e_2 = x_{j1}x_{j2}$, $1 \leq i \leq n - 1$ and $1 \leq j \leq n$. We will prove $\gamma_t(H) \leq \gamma_t(G_{n,2})$.

Consider $G_{i,2}$ and $H_{n-i,2}$. Then both $G_{i,2}$ and $H_{n-i,2}$ do not contain the edge e_1. There are several subcases.

If $i \equiv 2 \,(\mathrm{mod}\,3)$, then $n - i \equiv 2 \,(\mathrm{mod}\,3)$ since $n \equiv 1 \,(\mathrm{mod}\,3)$. By Lemma 3.4, $b_t(G_{i,2}) = 2 = b_t(H_{n-i,2})$, which implies $\gamma_t(G_{i,2}) = \gamma_t(G_{i,2} - e_2)$ if e_2 is in $G_{i,2}$, and $\gamma_t(H_{n-i,2}) = \gamma_t(H_{n-i,2} - e_2)$ if e_2 is in $H_{n-i,2}$. No matter which case arises, by Lemma 2.1, we have

$$\begin{aligned} \gamma_t(H) &\leq \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2}) \\ &= 2\left\lfloor \frac{i+2}{3} \right\rfloor + 2\left\lfloor \frac{n-i+2}{3} \right\rfloor \\ &\leq 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}). \end{aligned} \tag{3.6}$$

If $i \equiv 1 \,(\mathrm{mod}\,3)$ and $j \leq i$, then e_2 is in $G_{i,2}$. By (3.5), $\gamma_t(G_{i,2} - e_2) \leq \gamma_t(G_{i,2})$. Thus, the inequalities (3.6) hold.

If $i \equiv 0 \,(\mathrm{mod}\,3)$ and $j \geq i + 1$, then $n - i \equiv 1 \,(\mathrm{mod}\,3)$ and e_2 is in $H_{n-i,2}$. Since $H_{n-i,2} \cong G_{n-i,2}$, by (3.5), we have $\gamma_t(H_{n-i,2} - e_2) \leq \gamma_t(H_{n-i,2})$. Thus, the inequalities (3.6) hold.

The remainder is the case either $i \equiv 1 \,(\mathrm{mod}\,3)$ and $j \geq i + 1$ or $i \equiv 0 \,(\mathrm{mod}\,3)$ and $j \leq i$. Essentially, the two cases are the same by replacing $n - i$ for i. We only consider the latter case, that is, $i \equiv 0 \,(\mathrm{mod}\,3)$ and $j \leq i$.

If $j = i$, let $D = \{x_{kl} : k \equiv 2 \,(\mathrm{mod}\,3), l = 1, 2\} \cup \{x_{n1}, x_{n2}\}$, then D is a total dominating set of H, and so,

$$\gamma_t(H) \leq |D| = 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}).$$

We now assume $j < i$. Consider $G_{i-1,2}$ and $H_{n-(i+1),2}$. Let D' be a minimum total dominating set of $G_{i-1,2} - e_2$, and D'' be a minimum total dominating set of $H_{n-(i+1),2}$. Then $D = D' \cup D'' \cup \{x_{i2}, x_{(i+1)2}\}$ is a total dominating set of H. Since $i - 1 \equiv 2 \,(\mathrm{mod}\,3)$, $\gamma_t(G_{i-1,2}) = \gamma_t(G_{i-1,2} - e_2)$ by Lemma 3.4. $H_{n-(i+1),2}$ contains neither e_1 nor e_2. By Lemma 2.1, we have

$$\begin{aligned} \gamma_t(H) &\leq |D| = |D'| + |D''| + 2 \\ &\leq 2 + \gamma_t(G_{i-1,2}) + \gamma_t(H_{n-(i+1),2}) \\ &= 2 + 2\left\lfloor \frac{i+1}{3} \right\rfloor + 2\left\lfloor \frac{n-i+1}{3} \right\rfloor \\ &= 2\left\lfloor \frac{n+2}{3} \right\rfloor = \gamma_t(G_{n,2}). \end{aligned}$$

Case 3 Both e_1 and e_2 are horizontal edges.

Without loss of generality, let $e_1 = x_{i1}x_{(i+1)1}$ and $e_2 = x_{kj}x_{(k+1)j}$ are two distinct horizontal edges, where $1 \leq j \leq 2$ and $i \leq k < n$, and $j = 2$ if $i = k$. To prove $\gamma_t(H) \leq \gamma_t(G_{n,2})$, we consider three subcases.

Subcase 3.1 $k = i$.

In this subcase, $e_2 = x_{i2}x_{(i+1)2}$, H is disconnected and has exact two connected components $G_{i,2}$ and $H_{n-i,2}$. Since both $G_{i,2}$ and $H_{n-i,2}$ contain neither of e_1 and e_2, we have $\gamma_t(H) = \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2}) = \gamma_t(G_{n,2})$ by Lemma 2.1.

Subcase 3.2 $k = i + 1$.

In this subcase, $G_{i,2}$ and $H_{n-i-1,2}$ contain neither e_1 nor e_2.

If $i \equiv 0$ or $1 \,(\text{mod}\,3)$, let D' be a minimum total dominating set of $G_{i-1,2}$, and D'' be a minimum total dominating set of $H_{n-i-1,2}$, then $D = D' \cup D'' \cup \{x_{i2}, x_{(i+1)2}\}$ is a total dominating set of H. Note $D' = \emptyset$ if $i = 1$. By Lemma 2.1, we have

$$
\begin{aligned}
\gamma_t(H) \ &\leq |D| \leq 2 + \gamma_t(G_{i-1,2}) + \gamma_t(H_{n-i-1,2}) \\
&= 2 + 2\lfloor \tfrac{i+1}{3} \rfloor + 2\lfloor \tfrac{n-i+1}{3} \rfloor \\
&= 2\lfloor \tfrac{n+2}{3} \rfloor = \gamma_t(G_{n,2}).
\end{aligned}
$$

If $i \equiv 2 \,(\text{mod}\,3)$, let D' be a minimum total dominating set of $G_{i-2,2}$, and D'' be a minimum total dominating set of $H_{n-i-2,2}$, then $D = D' \cup D'' \cup \{x_{(i-1)p}, x_{ip}, x_{(i+1)p}, x_{(i+2)p}\}$, where $p = 3 - j$, is a total dominating set of H. Note $D' = \emptyset$ if $i = 2$ and $D'' = \emptyset$ if $i = n - 2$. By Lemma 2.1, we have

$$
\begin{aligned}
\gamma_t(H) \ &\leq |D| \leq 4 + \gamma_t(G_{i-2,2}) + \gamma_t(H_{n-i-2,2}) \\
&= 4 + 2\lfloor \tfrac{i}{3} \rfloor + 2\lfloor \tfrac{n-i}{3} \rfloor \\
&= 2\lfloor \tfrac{n+2}{3} \rfloor = \gamma_t(G_{n,2}).
\end{aligned}
$$

Subcase 3.3 $k > i + 1$.

In this case, e_2 is in $H_{n-i,2}$.

If $i \equiv 2 \,(\text{mod}\,3)$, then $n - i \equiv 2 \,(\text{mod}\,3)$ as $n \equiv 1 \,(\text{mod}\,3)$. Thus, $b_t(H_{n-i,2}) = 2$ by Lemma 3.4, which implies $\gamma_t(H_{n-i,2} - e_2) = \gamma_t(H_{n-i,2})$. If $i \equiv 0 \,(\text{mod}\,3)$, then $n - i \equiv 1 \,(\text{mod}\,3)$. Since e_2 is a horizontal edge in $H_{n-i,2}$, by Subcase 3.1, we also have $\gamma_t(H_{n-i,2} - e_2) = \gamma_t(H_{n-i,2})$. Thus, when $i \not\equiv 1 \,(\text{mod}\,3)$, we have

$$
\begin{aligned}
\gamma_t(H) \ &\leq \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2} - e_2) \\
&= \gamma_t(G_{i,2}) + \gamma_t(H_{n-i,2}) \\
&= \gamma_t(G_{n,2})
\end{aligned}
$$

If $k \not\equiv 0 \,(\text{mod}\,3)$ then, by replacing $G_{k,2}$ and $H_{n-k,2}$ by $H_{n-i,2}$ and $G_{i,2}$, respectively, we still have

$$
\begin{aligned}
\gamma_t(H) \ &\leq \gamma_t(G_{k,2} - e_1) + \gamma_t(H_{n-k,2}) \\
&= \gamma_t(G_{k,2}) + \gamma_t(H_{n-k,2}) \\
&= \gamma_t(G_{n,2}).
\end{aligned}
$$

Now, we assume $i \equiv 1 \,(\text{mod}\,3)$ and $k \equiv 0 \,(\text{mod}\,3)$. We consider three subgraphs $G_{i-1,2}$, $H_{n-(k+1),2}$ and $H_{k-1-(i+1),2}$.

Let D' be a minimum total dominating set of $G_{i-1,2}$, D'' be a minimum total dominating set of $H_{n-(k+1),2}$, and D''' be a minimum total dominating set of $H_{k-1-(i+1),2}$. Then $D = D' \cup D'' \cup D''' \cup \{x_{i2}, x_{(i+1)2}, x_{kp}, x_{(k+1)p}\}$, where $p = 3 - j$, is a total dominating set of H. Note $D' = \emptyset$ if $i = 1$, $D'' = \emptyset$ if $k = n - 1$ and $D''' = \emptyset$ if $k = i + 2$. By Lemma 2.1, we have

$$
\begin{aligned}
\gamma_t(H) \;&\leq |D| \leq 4 + \gamma_t(G_{i-1,2}) + \gamma_t(H_{n-k-1,2}) + \gamma_t(H_{k-i-2,2}) \\
&= 4 + 2\lfloor \tfrac{i+1}{3} \rfloor + 2\lfloor \tfrac{n-k-1+2}{3} \rfloor + 2\lfloor \tfrac{k-i-2+2}{3} \rfloor \\
&= 4 + 2\lfloor \tfrac{n-4}{3} \rfloor = 2\lfloor \tfrac{n+2}{3} \rfloor \\
&= \gamma_t(G_{n,2}).
\end{aligned}
$$

Summing up all cases, we prove the lemma. ∎

According to the above lemmas, we can state our results in this section as follows.

Theorem 3.1 *For any integer $n \geq 2$,*

$$
b_t(G_{n,2}) = \begin{cases} 1 \text{ if } n \equiv 0 \,(\mathrm{mod}\,3), \\ 2 \text{ if } n \equiv 2 \,(\mathrm{mod}\,3), \\ 3 \text{ if } n \equiv 1 \,(\mathrm{mod}\,3). \end{cases}
$$

4 The total bondage number of $G_{n,3}$

In this section, we will determine $b_t(G_{n,3}) = 1$ for $n \geq 2$. In this case, $Y_i = \{x_{ij}| \ 1 \leq j \leq 3\}$ for $1 \leq i \leq n$.

Lemma 4.1 *Let D be a minimum total dominating set of $G_{n,3}$. Then $|D \cap Y_1| \leq 2$ and $|D \cap Y_n| \leq 2$.*

Proof. Without loss of generality, we only show $|D \cap Y_n| \leq 2$. By contradiction, suppose that there exists a minimum total dominating set D of $G_{n,3}$ such that $|D \cap Y_n| = 3$. Then D is still a total dominating set of $G_{n+1,3}$. By Lemma 2.1, $n = \gamma_t(G_{n,3}) = |D| = \gamma_t(G_{n+1,3}) = n + 1$, a contradiction. Therefore, $|D \cap Y_n| \leq 2$. ∎

Lemma 4.2 *Let D be a total dominating set of $G_{n,3}$. If at least one of x_{n1} and x_{n3} is in D, then $|D| \geq n + 1$.*

Proof. Let D be a total dominating set of $G_{n,3}$.

We first consider that both x_{n1} and x_{n3} are in D. If $X_{n2} \in D$, then $|D \cap Y_n| = 3$. By Lemma 4.1, D is not a minimum total dominating set of $G_{n,3}$. Thus, by Lemma 2.1, we have $|D| \geq \gamma_t(G_{n,3}) + 1 = n + 1$.

Assume $x_{n2} \notin D$ below. Since x_{n2} is not in D, both $x_{(n-1)1}$ and $x_{(n-1)3}$ must be in D, which dominate x_{n1} and x_{n3}, respectively. Let $D' = (D \setminus \{x_{n1}, x_{n3}\}) \cup \{x_{(n-1)2}\}$. Then D' is still a total dominating set of $G_{n,3}$, and $|D'| < |D|$. By Lemma 2.1, $|D| \geq |D'| + 1 \geq \gamma_t(G_{n,3}) + 1 = n + 1$.

We now consider that only one of x_{n1} and x_{n3} is in D. Without loss of generality, we can assume $x_{n1} \in D$ and $x_{n3} \notin D$. We prove $|D| \geq n + 1$ by induction on n.

It is clear that $|D| \geq 3$ for $n = 2$. Suppose $|D| \geq k+1$ for any integer $k < n$. We prove that $|D| \geq n + 1$ for $n \geq 3$. We can assume that $|D \cap Y_i| \leq 2$ for each $i = 2, 3, \ldots, n - 1$

since if $|D \cap Y_i| = 3$ for some i with $2 \le i \le n-1$, then $(D \setminus \{x_{i1}, x_{i3}\}) \cup \{x_{(i-1)2}, x_{(i+1)2}\}$ is still a total dominating set of $G_{n,3}$ with the cardinality at most $|D|$.

If $x_{i2} \notin D$ for each $i = 2, 3, \ldots, n-1$, then each vertex in $D \setminus \{x_{n1}\}$ can totally dominate at most three vertices, and x_{n1} can dominate only two vertices. Thus, D can totally dominate at most $2 + 3(|D| - 1)$ vertices. On the other hand, D can totally dominate all $3n$ vertices. From the two facts, we can deduce $3n \le 2 + 3(|D| - 1)$, which yields $|D| \ge n + 1$.

Now assume $x_{i2} \in D$ for some i with $2 \le i \le n-1$. Let i_0 be the largest index such that $x_{i_0 2} \in D$ for $2 \le i_0 \le n-1$. If $x_{n2} \in D$ and $i_0 = n-1$, then $D \setminus \{x_{n1}\}$ is still a total dominating set of $G_{n,3}$. Thus, $|D| \ge \gamma_t(G_{n,3}) + 1 = n + 1$ by Lemma 2.1. We assume $i_0 \ne n-1$ if $x_{n2} \in D$ in the following discussion. There are two cases

Case 1 $D \cap Y_{i_0} = \{x_{i_0 2}\}$.

In this case, since $x_{(i_0+1)2} \notin D$ by the maximality of i_0, $x_{(i_0-1)2} \in D$, which dominates the vertex $x_{i_0 2}$. Let $D_1 = D \cap V(G_{i_0,3})$. Then D_1 is a total dominating set of $G_{i_0,3}$.
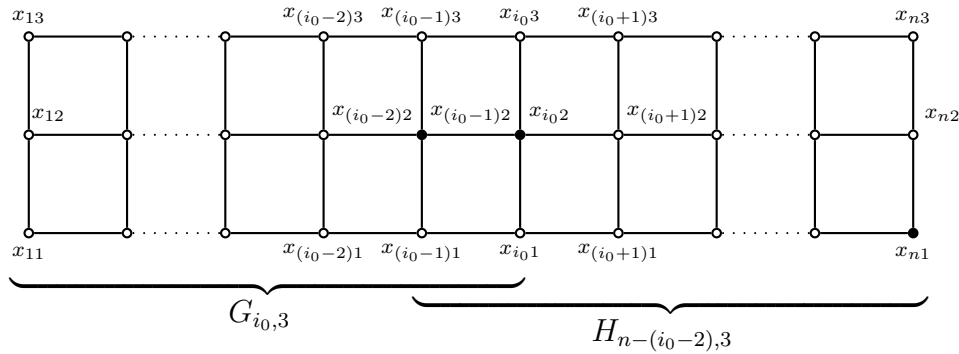

Figure 3: Two subgraphs $G_{i_0,3}$ and $H_{n-(i_0-2),3}$ of $G_{n,3}$

If $D \cap Y_{i_0-1} = \{x_{(i_0-1)2}\}$, then $D_2 = D \cap V(H_{n-(i_0-2),3})$ is a total dominating set of $H_{n-(i_0-2),3}$, and $|D_1 \cap D_2| = |\{x_{i_0-12}, x_{i_0 2}\}| = 2$ (see Figure 3). Since $H_{n-(i_0-2),3} \cong G_{n-(i_0-2),3}$ and D_2 satisfies the condition in the lemma (i.e., $x_{n1} \in D_2$), by the induction hypothesis $|D_2| \ge (n - i_0 + 2) + 1 = n - i_0 + 3$. By Lemma 2.1, $|D_1| \ge i_0$. Thus,

$$|D| \ge |D_1| + |D_2| - 2 \ge i_0 + n - i_0 + 3 - 2 = n + 1.$$

If $|D \cap Y_{i_0-1}| = 2$, then $D_3 = D \cap V(H_{n-(i_0-2),3})$ is a total dominating set of $H_{n-(i_0-2),3}$. Let $D_4 = D \cap V(G_{i_0-1}, 3)$. Then $|D_4 \cap D_3| = 2$. Since both of D_3 and D_4 satisfy the condition in the lemma, by the induction hypothesis, $|D_4| \ge i_0$ and $|D_3| \ge n - (i_0 - 2) + 1$. Thus,

$$|D| \ge |D_4| + |D_3| - 2 \ge i_0 + n - i_0 + 3 - 2 = n + 1.$$

Case 2 If $|D \cap Y_{i_0}| = 2$, then $D_5 = D \cap V(G_{i_0,3})$ is a total dominating set of $G_{i_0,3}$, $D_6 = D \cap V(H_{n-(i_0-1),3})$ is a total dominating set of $H_{n-(i_0-1),3}$ and $|D_5 \cap D_6| = 2$. Since both D_4 and D_5 satisfy the condition in the lemma, by the induction hypothesis, $|D_5| \ge i_0 + 1$ and $|D_6| \ge n - i_0 + 1 + 1 = n - i_0 + 2$. Thus,

$$|D| \ge |D_5| + |D_6| - 2 \ge i_0 + 1 + n - i_0 + 2 - 2 = n + 1.$$

The proof of the lemma is complete. ∎

Theorem 4.1 $b_t(G_{n,3}) = 1$ *for* $n \geq 2$.

Proof. Let $H = G_{n,3} - x_{(n-1)2}x_{n2}$ and D be a minimum total dominating set of H. Whether x_{n2} is in D or not, at least one of two vertices x_{n1} and x_{n3} is in D, which dominates x_{n2} in H. By Lemma 4.2, $|D| \geq n+1$. Combining this fact with Lemma 2.1, we have $\gamma_t(H) = |D| \geq n+1 = \gamma_t(G_{n,3}) + 1$. Therefore, $b_t(G_{n,3}) = 1$. ∎

5 The total bondage number of $G_{n,4}$

In this section, we determine the exact value of $b_t(G_{n,4})$ for $n \equiv 1, 4 \,(\mathrm{mod}\,5)$, and establish the upper bounds of $b_t(G_{n,4})$ for $n \equiv 0, 2, 3 \,(\mathrm{mod}\,5)$.

Lemma 5.1 $b_t(G_{n,4}) = 1$ *for* $n \geq 7$ *and* $n \equiv 1 \,(\mathrm{mod}\,5)$.

Proof. Let D be a minimum total dominating set of $G_{n,4} - x_{n2}x_{n3}$. It is easy to see that $|D \cap (Y_{n-1} \cup Y_n)| \geq 4$. Thus, $|D| \geq 4 + |D \cap V(G_{n-2,4})|$. When $n \geq 7$, $n - 3 \equiv 3 \,(\mathrm{mod}\,5)$ and $n - 3 \geq 4$. By Lemma 2.1, we have $\gamma_t(G_{n-3,4}) = \lfloor \frac{6(n-3)+8}{5} \rfloor + 1$. Thus, by Lemma 2.2, we have

$$
\begin{aligned}
|D \cap V(G_{n-2,4})| \;&\geq\; \gamma_t(G_{n-3,4}) \\
&= \left\lfloor \frac{6(n-3)+8}{5} \right\rfloor + 1 = \left\lfloor \frac{6n+8}{5} \right\rfloor - 2
\end{aligned}
$$

and, hence,

$$
\gamma_t(G_{n,4} - x_{n2}x_{n3}) = |D| \geq 2 + \left\lfloor \frac{6n+8}{5} \right\rfloor > \gamma_t(G_{n,4}).
$$

Therefore, $b_t(G_{n,4}) = 1$. ∎

To determine $b_t(G_{n,4})$ for $n \equiv 4 \,(\mathrm{mod}\,5)$, we state two simple observations, see Figure 4 for $n = 9$.
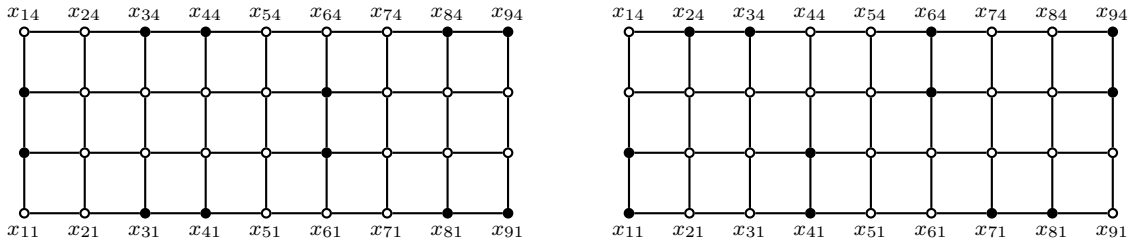

Figure 4: Two minimum total dominating sets (bold vertices) of $G_{9,4}$ defined in Proposition 5.1 and Proposition 5.2, respectively.

Proposition 5.1 *For* $n \equiv 4 \,(\mathrm{mod}\,5)$, *both*
$D = \{x_{i2}, x_{i3}, x_{(i+2)1}, x_{(i+3)1}, x_{(i+2)4}, x_{(i+3)4} : i \equiv 1 \,(\mathrm{mod}\,5), 1 \leq i \leq n-3\}$ *and*
$D' = \{x_{i2}, x_{i3}, x_{(i-3)1}, x_{(i-2)1}, x_{(i-3)4}, x_{((i-2))4} : i \equiv 4 \,(\mathrm{mod}\,5), 4 \leq i \leq n\}$
are minimum total dominating sets of $G_{n,4}$.

Proposition 5.2 *For $n \equiv 4 \,(\mathrm{mod}\,5)$, both*

$D = \{x_{i1}, x_{i2}, x_{(i+3)1}, x_{(i+3)2}, x_{(i+1)4}, x_{(i+2)4} : i \equiv 1\,(\mathrm{mod}\,10), 1 \leq i \leq n-3\}$

$\qquad \cup \{x_{j3}, x_{j4}, x_{(j+3)3}, x_{(j+3)4}, x_{(j+1)1}, x_{(j+2)1} : j \equiv 6\,(\mathrm{mod}\,10), 1 \leq j \leq n-3\}$ *and*

$D' = \{x_{i3}, x_{i4}, x_{(i+3)3}, x_{(i+3)4}, x_{(i+1)1}, x_{(i+2)1} : i \equiv 1\,(\mathrm{mod}\,10), 1 \leq i \leq n-3\}$

$\qquad \cup \{x_{j1}, x_{j2}, x_{(j+3)1}, x_{(j+3)2}, x_{(j+1)4}, x_{(j+2)4} : j \equiv 6\,(\mathrm{mod}\,10), 1 \leq j \leq n-3\}$

are minimum total dominating sets of $G_{n,4}$.

Lemma 5.2 $b_t(G_{n,4}) = 2$ *for $n \equiv 4\,(\mathrm{mod}\,5)$.*

Proof. We can direct check $b_t(G_{4,4}) = 2$, and assume $n \geq 9$ below. For any edge $e \in E(G_{n,4})$, it is easy to verify that D or D' defined in Proposition 5.1 or Proposition 5.2 is also a minimum total dominating set of $G_{n,4} - e$. Thus, $b_t(G_{n,4}) \geq 2$. We now prove that $b_t(G_{n,4}) \leq 2$.

Let $H = G_{n,4} - x_{(n-1)1}x_{n1} - x_{(n-1)2}x_{n2}$ and let S be a minimum total dominating set of H. Then the vertex x_{n2} must be in S to dominate x_{n1}, and at least one of x_{n1} and x_{n3} must be in S to dominate x_{n2} in H, that is, $|Y_n \cap S| \geq 2$.

If $|Y_n \cap S| \geq 3$ then $|S| \geq |S \cap V(G_{n-1,4})| + 3$. By Lemma 2.2 and Lemma 2.1, we have

$$
\begin{aligned}
|S \cap V(G_{n-1,4})| &\geq \gamma_t(G_{n-2,4}) \\
&= \left\lfloor \tfrac{6(n-2)+8}{5} \right\rfloor = \left\lfloor \tfrac{6n+8}{5} \right\rfloor - 2
\end{aligned}
$$

and, hence,

$$
\gamma_t(H) = |S| \geq \gamma_t(G_{n-2,4}) + 3 \geq 1 + \left\lfloor \frac{6n+8}{5} \right\rfloor > \gamma_t(G_{n,4}).
$$

If $|Y_n \cap S| = 2$, then $Y_n \cap S$ can totally dominate at most one vertex in $G_{n-1,4}$, that is, $x_{(n-1)3}$ if so. Thus, $(S \cap V(G_{n-1,4})) \cup \{x_{(n-1)2}\}$ is a total dominating set of $G_{n-1,4}$, which implies $|S| \geq |(S \cap V(G_{n-1,4})) \cup \{x_{(n-1)2}\}| + 1 \geq \gamma_t(G_{n-1,4}) + 1$. By Lemma 2.1, we have

$$
\gamma_t(H) = |S| \geq \gamma_t(G_{n-1,4}) + 1 > \gamma_t(G_{n,4}).
$$

Therefore, $b_t(G_{n,4}) \leq 2$. ∎

Lemma 5.3 $b_t(G_{n,4}) \leq 3$ *for $n \equiv 2\,(\mathrm{mod}\,5)$.*

Proof. $b_t(G_{4,2}) = 3$ by Theorem 3.1, and $b_t(G_{4,7}) \leq 3$ by checking direct. Assume $n \geq 12$ below. Let $H = G_{n,4} - x_{(n-1)1}x_{n1} - x_{(n-1)2}x_{n2} - x_{n2}x_{n3}$ and let S be a minimum total dominating set of H. Since $x_{n1}x_{n2}$ is an isolated edge in H, both x_{n1} and x_{n2} must be in S. To dominate the three vertices $x_{(n-1)1}$, x_{n3} and x_{n4}, we need at least three other vertices in S. In other words, $|(Y_{n-2} \cup Y_{n-1} \cup Y_n) \cap S| \geq 5$. Thus,

$$
|S| \geq 5 + |S \cap V(G_{n-3,4})|. \tag{5.1}
$$

By Lemma 2.2 and Lemma 2.1, when $n \geq 8$ and $n \equiv 2\,(\mathrm{mod}\,5)$, we have

$$
\begin{aligned}
|S \cap V(G_{n-3,4})| &\geq \gamma_t(G_{n-4,4}) \\
&= \left\lfloor \tfrac{6(n-4)+8}{5} \right\rfloor + 1 = \left\lfloor \tfrac{6n+8}{5} \right\rfloor - 4,
\end{aligned}
$$

that is,

$$
|S \cap V(G_{n-3,4})| \geq \left\lfloor \frac{6n+8}{5} \right\rfloor - 4. \tag{5.2}
$$

It follows from (5.1), (5.2) and Lemma 2.1 that

$$\gamma_t(H) = |S| \geq 1 + \left\lfloor \frac{6n+8}{5} \right\rfloor > \gamma_t(G_{n,4}).$$

Therefore, $b_t(G_{n,4}) \leq 3$. ∎

Lemma 5.4 $b_t(G_{n,4}) \leq 4$ for $n \equiv 0, 3 \, (\mathrm{mod}\, 5)$.

Proof. By direct checking we have $b_t(G_{5,4})) \leq 4$ and $b_t(G_{8,4})) \leq 4$. Assume $n \geq 10$ below. Let $H = G_{n,4} - x_{(n-5)1}x_{(n-4)1} - x_{(n-5)2}x_{(n-4)2} - x_{(n-5)3}x_{(n-4)3} - x_{(n-5)4}x_{(n-4)4}$ and let S be a minimum total dominating set of H. Then H consists of two grid subgraphs $G_{n-5,4}$ and $H_{5,4}$, and so

$$|S| = \gamma_t(G_{n-5,4}) + \gamma_t(H_{5,4}). \tag{5.3}$$

Since $n - 5 \equiv 0, 3 \, (\mathrm{mod}\, 5)$ and $5 \equiv 0 \, (\mathrm{mod}\, 5)$, by Lemma 2.1, we have

$$\begin{aligned} \gamma_t(G_{n-5,4}) + \gamma_t(H_{5,4}) &= \left\lfloor \frac{6(n-5)+8}{5} \right\rfloor + 1 + 8 \\ &= \left\lfloor \frac{6n+8}{5} \right\rfloor + 3 > \gamma_t(G_{n,4}). \end{aligned} \tag{5.4}$$

Combining (5.3) with (5.4), we have

$$\gamma_t(H) = \gamma_t(G_{n-5,4}) + \gamma_t(H_{5,4}) > \gamma_t(G_{n,4}).$$

Therefore, $b_t(G_{n,4}) \leq 4$. ∎

Summing up the above lemmas, we can state our result, in this section, as follows.

Theorem 5.1 For any integer $n \geq 1$, $b_t(G_{6,4}) = 2$, and

$$b_t(G_{n,4}) \begin{cases} = 1 & \text{if } n \equiv 1 \, (\mathrm{mod}\, 5) \text{ and } n \neq 6; \\ = 2 & \text{if } n \equiv 4 \, (\mathrm{mod}\, 5); \\ \leq 3 & \text{if } n \equiv 2 \, (\mathrm{mod}\, 5); \\ \leq 4 & \text{if } n \equiv 0, 3 \, (\mathrm{mod}\, 5). \end{cases}$$

6   Concluding Remarks

In this paper, we investigate the total bondage number $b_t(G_{n,m})$ of an (n,m)-grid graph $G_{n,m}$ for $2 \leq m \leq 4$, completely determine the exact values of $b_t(G_{n,2})$ and $b_t(G_{n,3})$. We also partially determine the exact values of $b_t(G_{n,4})$, and establishes the upper bounds of $b_t(G_{n,4})$ for otherwise. We have attempted to decrease the two upper bounds given in Theorem 5.1 for $n \equiv 2 \, (\mathrm{mod}\, 5)$ and $n \equiv 0, 3 \, (\mathrm{mod}\, 5)$ or to prove that they are tight when n is large enough, but we have not been able to bring home the bacon. Noting the two upper bounds are tight for some small n's, we guess that the two upper bounds are tight for $n \geq 7$. To prove this conjecture, it may be necessary to find a new method since, according to our way, the removal of any three edges results in many complicated cases. We also have tried to discuss $b_t(G_{n,m})$ for general n and m, but it strongly depends on the value of $\gamma_t(G_{n,m})$, which has not been determined as yet. Thus, it may also be necessary to determine the value of $\gamma_t(G_{n,m})$ for general n and m. These questions are our further work.

Acknowledgements

References

[1] E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi, Total domination in graphs. Networks, 10 (1980), 211-219.

[2] K. Carlson and M. Develin, On the bondage number of planar and directed graphs. Discrete Math., 306 (8-9) (2006), 820-826.

[3] J. E. Dunbar, T. W. Haynes, U. Teschner and L. Volkmann, Bondage, insensitivity, and reinforcement. Domination in Graphs: Advanced Topics (T. W. Haynes, S. T. Hedetniemi, P. J. Slater eds.), Monogr. Textbooks Pure Appl. Math., 209, Marcel Dekker, New York, 1998, pp. 471-489.

[4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph. Discrete Math., 86 (1990), 47-57.

[5] M. Fischermann, D. Rautenbach and L. Volkmann, Remarks on the bondage number of planar graphs. Discrete Math., 260 (2003), 57-67.

[6] S. Gravier, Total domination number of grid graphs. Discrete Applied Mathematics, 121 (2002), 119-128.

[7] B. L. Hartnell, L. K. Jorgensen, P. D. Vestergaard and C. Whitehead, Edge stability of the k-domination number of trees. Bulletin of the Institute of Combinatorics and its Applications, 22 (1998), 31-40.

[8] B. L. Hartnell and D. F. Rall, A characterization of trees in which no edge is essential to the domination number. Ars Combinatoria, 33 (1992), 65-76.

[9] B. L. Hartnell and D. F. Rall, Bounds on the bondage number of a graph. Discrete Mathematics, 128 (1994), 173-177.

[10] J. H. Hattingh and A. R. Plummer, Restrained bondage in graphs. Discrete Math., 308 (23) (2008), 5446-5453.

[11] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*. Marcel Dekker, New York, 1998.

[12] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds.), *Domination in Graphs: Advanced Topics*. Marcel Dekker, New York, 1998.

[13] M. A. Henning, A survey of selected recent results on total domination in graphs. Discrete Mathematics, 309(1) (2009), 32-63.

[14] F.-T. Hu and J.-M. Xu, Complexity of Bondage and Reinforcement. A manuscript submitted to Discrete Applied Mathematics, 2010.

[15] J. Huang and J.-M. Xu, The bondage numbers of extended de Bruijn and Kautz digraphs. Computer and Mathematics with Applications, 51(6-7) (2006), 1137-1147.

[16] J. Huang and J.-M. Xu, The bondage number of graphs with small crossing number. Discrete Math., 307(15) (2007), 1881-1897.

[17] J. Huang and J.-M. Xu, The total domination and bondage numbers of extended de bruijn and Kautz digraphs. Computer and Mathematics with Applications, 53(8) (2007), 1206-1213.

[18] J. Huang and J.-M. Xu, The bondage numbers and efficient dominations of vertex-transitive graphs. Discrete Math., 308(4) (2008), 571-582.

[19] L.-Y Kang, M. Y. Sohn and H. K. Kim, Bondage number of the discrete torus $C_n \times C_4$. Discrete Math., 303 (2005), 80-86.

[20] L. Kang and J. Yuan, Bondage number of planar graphs, Discrete Math., 222 (2000), 191-198.

[21] V. R. Kulli, D. K. Patwari, The total bondage number of a graph, in: Advances in Graph Theory (V. R. Kulli Ed.). Vishwa, Gulbarga, 1991, 227-235.

[22] H. Liu and L. Sun, The bondage and connectivity of a graph, Discrete Math., 263 (2003), 289-293.

[23] M. Y. Sohn, X.-D. Yuan and H. S. Jeong, The bondage number of $C_3 \times C_n$. J. Korean Math. Soc. 44(6) (2007), 1213-1231.

[24] N. Sridharan, M. D. Elias, V. S. A. Subramanian, Total bondage number of a graph. AKCE Int. J. Graphs Combin. 4 (2)(2007), 203-209.

[25] U. Teschner, New results about the bondage number of a graph. Discrete Math., 171 (1997), 249-259.

[26] J. Topp and P. D. Vestergaard, α_k and γ_k-stable graphs. *Discrete Mathematics*, **212** (2000), 149-160.

[27] J.-M. Xu, Theory and Application of Graphs. Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.